Multivariate Differentiation & Integral Calculus

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- e.g.
  - $f(x, y, z) = 3xy - y^2x + 2$
  - $f(\mathbf{x}) = f(x_1, \cdots, x_5) = x_1x_3 - x_2x_5$

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- Higher-order partial derivatives

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Partial Derivatives: Examples

- Find all the partial derivatives of

$$f(x, z) = xz$$

Partial Derivatives: Examples

- $f(x_1, x_2, x_3) = 6 + 3x_1 + \frac{5}{2}x_2 + x_3^2$

Partial Derivatives: Examples

- $f(x, z) = 3z^3 - 3z^2 + \sqrt{z} + x$

Partial Derivatives: Examples

- In a regression context, let

$$y = \beta_0 + \beta_1 x + \beta_2 z + \beta_3 xz + e,$$

find $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial z}$

Gradient Vector and Hessian Matrix

For multivariate function $f(x_1, \cdots, x_n)$

- **Gradient**: The (row) vector of first-order partial derivatives

$$\nabla f \equiv \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} & \cdots \end{bmatrix} \in \mathbb{R}^{1\times n}$$

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Gradient and Hessian: Example

- Let $f(x, y) = x^3y^4 + e^x - \ln(y)$
  - Find

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial x \, \partial y}$$

  - Find gradient of f
  - Find Hessian of f

Taylor Approximation

- Taylor Approximation: Linear approximation of a function around point $x = a$

$$
\begin{aligned}
f(x) &\approx f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \cdots \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n
\end{aligned}
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where $\nabla f(\mathbf{a})$ and $\mathbf{H}(\mathbf{a})$ are the gradient and Hessian evaluated at the vector $\mathbf{a}$

Why convex implies local minimum?

- By Taylor approximation of univariate function

$$
\begin{aligned}
f(x) - f(a) &\approx f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 \\
&= 0 \cdot (x-a) + \frac{1}{2} f''(a)(x-a)^2 \\
&\begin{cases} > 0 \text{ if } f''(a) > 0 \ \ (\text{so } f''(a) > 0 \Rightarrow f(x) > f(a) \Rightarrow f(a) \text{ min}) \\ < 0 \text{ if } f''(a) < 0 \ \ (\text{so } f''(a) < 0 \Rightarrow f(x) < f(a) \Rightarrow f(a) \text{ max}) \end{cases}
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- Similarly, for multivariate functions we want

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\begin{aligned}
f(\mathbf{x}) - f(\mathbf{a}) &\approx \nabla f(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \frac{1}{2}(\mathbf{x}-\mathbf{a})^\top \mathbf{H}(\mathbf{a})(\mathbf{x}-\mathbf{a}) \\
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Positive and Negative Definite Matrix

Definition

Consider a $n \times n$ matrix $\mathbf{A}$. If, for all $\mathbf{x} \in \mathbb{R}^n$ where $\mathbf{x} \neq \mathbf{0}$:

$$\mathbf{x}^\top \mathbf{A} \mathbf{x} > 0 \text{, we say } \mathbf{A} \text{ is \textbf{positive definite}};$$
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Property

For a 2×2 matrix $\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$:

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- Compare to values at the boundary (if solving global extremum)

Multivariate Optimization: Example

- Find the local extremum of

$$f(x_1, x_2) = 3(x_1 + 2)^2 + 4(x_2 + 4)^2$$

Multivariate Optimization: Example

- Suppose legislators are considering legislation $\mathbf{x} \in \mathbb{R}^2$. And suppose legislator i has utility function $U_i : \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$U_i(\mathbf{x}) = -(x_1 - \mu_1)^2 - (x_2 - \mu_2)^2.$$

What is legislator i's optimal policy?

Why study integral calculus?

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  - If $f'(x) = 2$, what is $f(x)$?
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  - A concrete example

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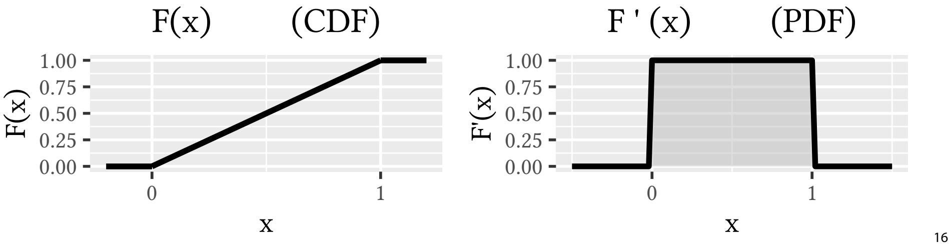

Antiderivative

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The antiderivative of a function f is a **function** whose derivative is f.
We often denote the antiderivative of f as F, i.e,

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  - $f(x) = 3e^{3x}$
- How do we write the antiderivative in a more systematic way?

Indefinite Integral

Definition

The antiderivative of $f(x)$ can also be written as

$$F(x) = \int f(x)\, dx,$$

which is called the indefinite integral. Thus, we have that

$$\frac{d\, F(x)}{d\, x} = \frac{d}{d\, x} \int f(x)\, dx = f(x).$$

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- Useful heuristic: $\frac{d}{d\,x}$ and $\int\ dx$ cancels out
  - Differentiation and integration are doing the reverse works

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which is called the indefinite integral. Thus, we have that

$$\frac{d\,F(x)}{d\,x} = \frac{d}{d\,x} \int f(x)\,dx = f(x).$$

- Useful heuristic: $\frac{d}{d\,x}$ and $\int\ dx$ cancels out
  - Differentiation and integration are doing the reverse works
- Some would also denote that (useful for probability, e.g. $F(x)$ is the CDF of X)

$$d\,F(x) = f(x)\,dx$$

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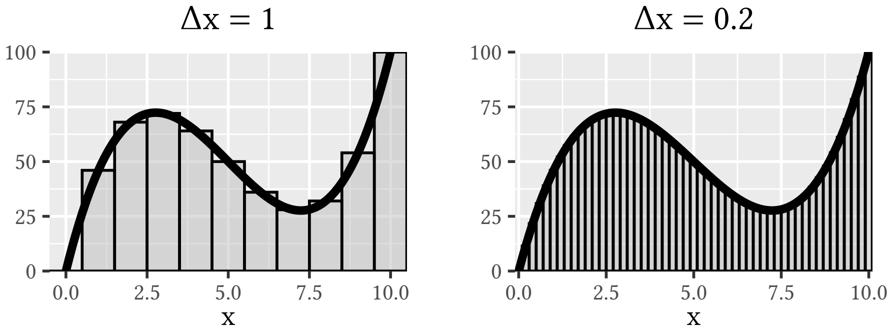
- Then the total area is $\sum_i f(x_i)\Delta x$



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- Definite integral: area from $x = a$ to $x = b$, which is a fixed number

$$\int_a^b f(x)\,dx$$

Fundamental Theorem of Calculus

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Let $f(x)$ be continuous over an interval $[a, b]$.

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- Part I. tells you a way to define the antiderivative $F(x)$
- Part II. tells you how to evaluate the definite integral: plug-in to $F(x)$

Fundamental Theorem of Calculus: Example

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- $F(x)\Big|_{x=1}^{2} = F(2) - F(1) = \left[\frac{1}{3}(2)^3 + C\right] - \left[\frac{1}{3}(1)^3 + C\right] = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$

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4. Areas can be combined as long as limits are linked:
$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

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$$\int \ln(x)\,dx = x\ln(x) - x + C \qquad \text{(Logarithm)}$$

Integration: Examples

- Find

$$\int x^3 dx$$

Integration: Examples

- Find

$$\int \frac{1}{x^5} dx$$

Integration: Examples

- Find

$$\int \sqrt{x}dx$$

Integration: Examples

- Find

$$\int_1^4 (2x + 1)dx$$

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- Useful for composite functions (square root, fractions, power, etc)

Integration by Substitution: Example

- Integrate $g(x) = x^2\sqrt{x+1}$
  - Let $u = x + 1$, then $du = dx$, substitute into $\int g(x)\,dx$

Integration by Substitution: Example

- Integrate $g(x) = \frac{1}{\sqrt{2\pi}} x e^{-\frac{x^2}{2}}$
  - Let $u = -\frac{x^2}{2}$, then $du = -x\,dx$, substitute into $\int g(x)\,dx$

Integration by Substitution: Example

- Find

$$\int x^4 e^{x^5}\, dx$$

  - Let $u = x^5$, then $du = 5x^4\, dx$

Integration by Substitution: Example

- Show that $\int a^x dx = \frac{a^x}{\ln(a)} + C$
  - Again, $a^x = e^{\ln a^x} = e^{x \ln a}$
  - Let $u = x \ln a$, then $du = \ln a \, dx$

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- Useful for products of x and e^x or $\ln x$

Integration by Parts: Examples

- Find
$$\int xe^x\,dx \quad \text{and} \quad \int_1^4 xe^x\,dx$$
    - Let $u = x$ and $dv = e^x\,dx$

Integration by Parts: Examples

- Show that $\int \ln x \, dx = x \ln x - x + C$
  - Let $u = \ln x$ and $dv = \frac{1}{x} \, dx$

- Show that $\int \log_a(x) \, dx = \frac{x \ln(x) - x}{\ln(a)} + C$
  - Again, $\log_a(x) = \frac{\ln x}{\ln a}$

Integration by Parts: Harder Examples

- Find
$$\int x^n e^{ax}\, dx$$
  - Let $u = x^n$ and $dv = e^{ax}\, dx$

Integration by Parts: Harder Examples

- Find
$$\int x^3 e^{-x^2}\,dx$$
  - Let $u = x^2$ and $dv = xe^{-x^2}\,dx$
  - Use substitution $t = -x^2$ on dv to get v

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- Many applications (infinite time-horizon bargaining, exponential distribution, etc)

Improper Integral: Example

- Let β be a fixed constant. Find

$$\int_0^\infty \frac{1}{\beta} e^{-\frac{x}{\beta}}\, dx$$

  - Note: This is the probability density function of the exponential distribution