Differential Calculus

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Why study differential calculus?

- How would you approximate $f(x)$ by a linear function around the point $x = a$?

$$f(x) \approx p(x-a) + q$$

  - We can let $q = f(a)$, then $p = \frac{f(x)-f(a)}{x-a}$ is the slope
  - To make it accurate, we should let x very close to a
- How would you find the maximum or minimum of a function?
- What is the shape of a function?

Functions

Definition

Intuitively, a function is a mapping from an input to a **unique** output.
Specifically, a function $f : X \to Y$ is a relation that associates **each element** x in a set X to a **single element** y in another set Y, denoted by

$$y = f(x).$$

X is called the Domain of f, Y is called the Codomain of f.
The Range is the subset of Y where f is defined, that is,

$$\text{Range}(f) \equiv f(X) = \{f(x) \,|\, x \in X\} \subseteq Y.$$

- Note: f is a function, $f(x)$ is the value of the function evaluate at x
- eg. $f(x) = x^2$, $f : \mathbb{R}$ (Domain) $\to \mathbb{R}$ (Codomain), $\text{Range}(f) = [0, \infty)$
- eg. $f(x) = \pm x$, f is not a function

Functions: Examples

- $f(x) = x + 1$
- $f(x) = 1/x$
- $f(x, y) = x^2 + y^2$
- $f(x) = \sin(x)$
- $f(x) = \sqrt{x}$
- $f(x) = \frac{3}{1+x^2}$
- Exercise: find their domain and range

Composite Functions

Definition

A composite function is a function of function.

Specifically, suppose $f : A \to B$ and $g : B \to C$.

Define the composite function $h \equiv g \circ f$, where $h : A \to C$ as

$$h(x) = (g \circ f)(x) = g(f(x))$$

- $f(x) = \sqrt{x}$, $g(x) = e^x$, $g(f(x)) = e^{\sqrt{x}}$, $f(g(x)) = \sqrt{e^x}$
- $f(x) = x$, $g(x) = x^2$, $g(f(x)) = x^2$, $f(g(x)) = x^2$
- $f(x) = 2^x$, $g(x) = \log_2(x)$, $g(f(x)) = x$, $f(g(x)) = x$
- $f(x) = \sqrt{x}$, $g(x) = x^2$, $g(f(x)) = x$, $f(g(x)) = |x|$

Inverse Functions

Definition

Suppose a function f is 1-1 (distinct inputs maps to distinct outputs).
The function g is the inverse of f, if their composite function maps back to itself, ie,

$$g(f(x)) = x.$$

We often denote $g \equiv f^{-1}$.

- $f(x) = 2x$, $g(x) = \frac{1}{2}x$ is the inverse function of f
- $f(x) = x^2$, its inverse is $\pm\sqrt{x}$ but this is **not** a function
- $f(x) = 2^x$, $g(x) = \log_2(x)$ is the inverse function of f

Limits of Functions

Definition

If a function $f(x)$ tends to L at point x_0 we say it has a limit L at x_0. Formally, let $f(x)$ be defined at each point around x_0. Then

$$\lim_{x \to x_0} f(x) = L \quad \text{or, equivalently,} \quad f(x) \to L \text{ as } x \to x_0$$

if for any (small positive) number ε, there exists a corresponding number $\delta > 0$ such that if $0 < |x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$.

- Note: This does **not** imply $f(x_0) = L$!!
- $\lim_{x \to 3} x^2 = 9$
- $\lim_{x \to \infty} \frac{1}{x} = 0$
- $\lim_{x \to \infty} 2^x = \infty$
- When in doubt, plot the function!

Properties of Limit

Property

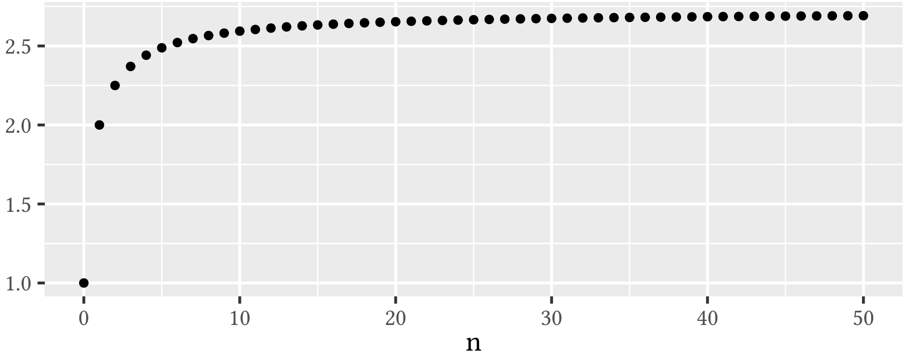
Let f and g be functions with $\lim_{x \to c} f(x) = K$ and $\lim_{x \to c} g(x) = L$. We have that

1. $\lim_{x \to c} \alpha f(x) = \alpha \lim_{x \to c} f(x) = \alpha K$
2. $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = K + L$
3. $\lim_{x \to c} f(x)g(x) = \left[\lim_{x \to c} f(x)\right] \cdot \left[\lim_{x \to c} g(x)\right] = KL$
4. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{K}{L}$, provided $L \neq 0$

- Note: K and L have to be real numbers, not $\pm\infty$

The Number e: Base Rate of Growth

$$f(n) = \left(1 + \frac{1}{n}\right)^n$$



- $f(n) = \left(1 + \frac{1}{n}\right)^n \to 2.7182818284... \equiv e$ as $n \to \infty$
- Furthermore, we have that $\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n = e^r$ (fix r, take $m = \frac{n}{r} \to \infty$ as $n \to \infty$)

Left and Right Limits

Definition

If x approaches x_0 from the right, we write $\lim_{x \to x_0^+} f(x) = L^+$.

If x approaches x_0 from the left, we write $\lim_{x \to x_0^-} f(x) = L^-$.

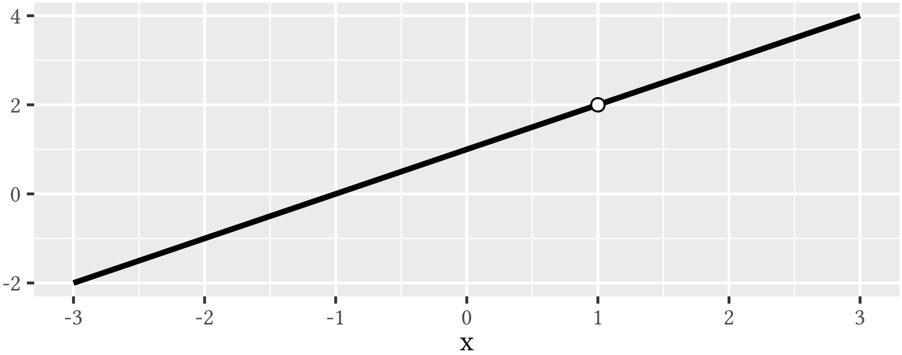
Theorem

$\lim_{x \to x_0} f(x) = L$ if and only if $L^+ = L^- (= L)$.

- $\lim_{x \to 0^+} \frac{1}{x} = \infty$, $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

Limits: An Example

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = \begin{cases} x + 1 & \text{if } x \neq 1; \\ \text{undefined} & \text{if } x = 1. \end{cases}$$



- $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 2$, but $f(1) \neq 2$!
- $f(1)$ is undefined; $f(x)$ is **discontinued** at $x = 1$

Continuity

Definition

A function f is continuous **at** x_0 if and only if

1. $\lim_{x \to x_0} f(x)$ exists, and
2. $\lim_{x \to x_0} f(x) = f(x_0)$.

In other words, the limiting value equals to the value of the function evaluate at that point.

If f is continuous **at all points** of $x \in X$, we say that f is continuous (on X).

- Continuity ensures that $f(x_0) = L$
- If f is continuous at c, can plug in to get the limit as $f(c)$

Rate of Change

Let's measure the rate of change of $f(x)$ at a point x_0 with a function $R(x)$:

$$R(x) = \frac{f(x) - f(x_0)}{x - x_0} = \frac{\Delta f}{\Delta x}$$

- Nominator: change in f
- Denominator: change in x
- $R(x)$ defines the rate of change
- A derivative will examine what happens with a small perturbation at x_0

Derivative at a Point and Differentiability

Definition

The limit of the rate of change $R(x)$ is the derivative of $f(x)$.

In other words,

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \equiv f'(x_0) \equiv \frac{df}{dx}(x_0)$$

is the derivative of f at x_0.

If this limit exists, we say that f is differentiable **at** x_0.

If f is differentiable **at all points** of $x \in X$, we say that f is differentiable (on X).

- $f(x) = x^2$, $x_0 = 1$, $f'(1) = \lim_{x \to 1} \frac{x^2-1}{x-1} = 2$
- $f(x) = |x|$, $x_0 = 0$, $f'(0) = \lim_{x \to 0} \frac{|x|}{x}$ is undefined (right limit 1, left limit -1)
- $f(x) = |x|$ is continuous but not differentiable (rate of change too abrupt)

Derivative as a Function

Definition

Suppose f is differentiable for all $x \in X$.
The derivative of the function $f(x)$ is defined by

$$f'(x) \equiv \frac{df}{dx}(x) \equiv \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

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- Note: f' and $\frac{df}{dx}$ are themselves **functions**
- If f is differentiable, we can find f' first and plug-in to get $f(x_0)$
- For a line, the derivative is the slope
- For a curve, the derivative is the slope of the line tangent to the curve at each x

Calculating Derivatives

- Let's first do some examples:
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- Rarely will we take limit to calculate derivative
- Rather, rely on **rules and properties** of derivatives
- Important: Do not forget core **intuition**
- Strategy: Work on problems

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$$f(x) = e^x \qquad f'(x) = e^x$$ (Exponential)

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$f(x) = \cos(x)$	$f'(x) = -\sin(x)$	

How to Do Operations

Algebra of Differentiation

Suppose f and g are both differentiable.

$$h(x) = cf(x) \qquad h'(x) = cf'(x) \qquad \text{(Constant)}$$

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$$h(x) = f(x)g(x) \qquad h'(x) = f'(x)g(x) + g'(x)f(x) \qquad \text{(Product)}$$

$$h(x) = \frac{f(x)}{g(x)} \qquad h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \qquad \text{(Quotient)}$$

Differentiation: Examples

- Find the derivative of $f(x) = (x^3)(2x^4)$

Differentiation: Examples

- Find the derivative of $f(x) = \frac{x^2+1}{x^2-1}$

Differentiation: Examples

- Show that $(x^k)' = k \cdot x^{k-1}$
  - Hint: By induction, suppose holds for $k - 1$, show holds for k

Differentiation: Examples

- Show that $(\log_a(x))' = \frac{1}{x \ln(a)}$
  - Hint: $\log_a(x) = \frac{\ln x}{\ln a}$ (since if $y = \log_a(x)$, $a^y = x$, take $\ln$)

Chain Rule: Derivative of Composite Functions

Chain Rule

Suppose both f and g are differentiable. The derivative of $(f \circ g)(x) \equiv f[g(x)]$ is

$$\frac{d}{dx}(f[g(x)]) = f'[g(x)]g'(x).$$

Or, equivalently,

$$(f(g(x)))' = f'(g(x))g'(x).$$

- Intuitively, we can think of f as a function of g and g as a function of x and write

$$\frac{df(g(x))}{dx} = \frac{df(g)}{dg} \cdot \frac{dg(x)}{dx}, \quad \text{or,} \quad \frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

- x changes f indirectly: First x affects g by $\frac{dg}{dx}$, then g affects f by $\frac{df}{dg}$

Chain Rule: Examples

- Find dy/dx for $y = (3x^2 + 5x - 7)^6$
  - Hint: Let $f(z) = z^6$ and $z = g(x) = 3x^2 + 5x - 7$

Chain Rule: Examples

- Find dy/dx for $y = \sin(x^3 + 4x)$
  - Hint: Let $f(z) = \sin(z)$ and $z = g(x) = x^3 + 4x$

Chain Rule: Examples

- Show that $(a^x)' = a^x(\ln(a))$
  - Hint: $a^x = e^{\ln a^x} = e^{x \ln a}$ (very important substitution)
  - Note: $(a^x)' = c \cdot a^x$ suggests exponential function is proportional to its own derivative!
  - e is the base a such that the proportion c is 1, and this is precisely why $(e^x)' = e^x$

Chain Rule: Examples

- Show the Generalized Power Rule:

$$\text{Let } y = [g(x)]^n, \quad \text{then } \frac{dy}{dx} = n[g(x)]^{n-1}g'(x)$$

Chain Rule: Examples

- Show that

$$\left(e^{u(x)}\right)' = e^{u(x)}u'(x)$$

Chain Rule: Examples

- Show that, for $u(x) > 0$,

$$(\ln u(x))' = \frac{u'(x)}{u(x)}$$

Higher-Order Derivatives

- What about the derivative of $f'(x)$ with respect to x?

$$f''(x) \equiv f^{(2)}(x) \equiv \frac{d}{dx}\left(\frac{df}{dx}\right)(x) \equiv \frac{d^2 f}{dx^2}(x) \equiv \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

- We can similarly define the derivatives of $f''(x)$, and so on

Higher-Order Derivatives: Example

- $f(x) = x^3$, find f', $f^{(2)}$, $f^{(3)}$, $f^{(4)}$

Increasing or Decreasing Functions

- Derivatives inform us about the shape of a function
- The first derivative, $f'(x)$, identifies whether the function $f(x)$ at the point x is increasing or decreasing at x
  - $f'(x) > 0 \Rightarrow$ Increasing
  - $f'(x) < 0 \Rightarrow$ Decreasing
  - $f'(x) = 0 \Rightarrow$ Neither increasing nor decreasing (i.e. max, min, or saddle point)

Increasing or Decreasing Functions: Example

- Determine where the function increase or decrease

$$f(x) = x^2$$

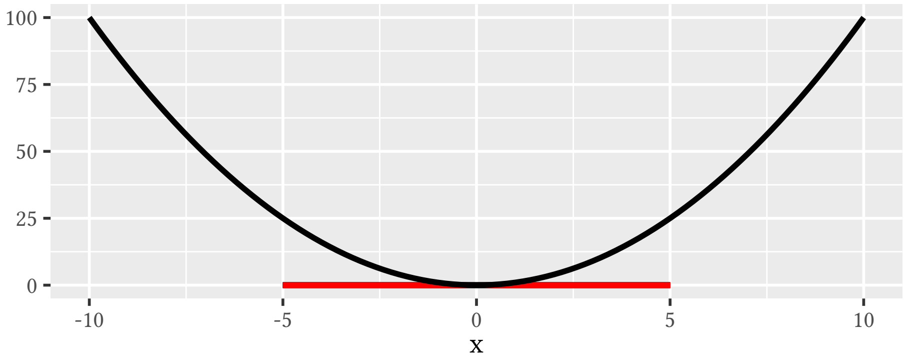
Increasing or Decreasing Functions: Example

- Determine where the function increase or decrease

$$f(x) = x^3$$

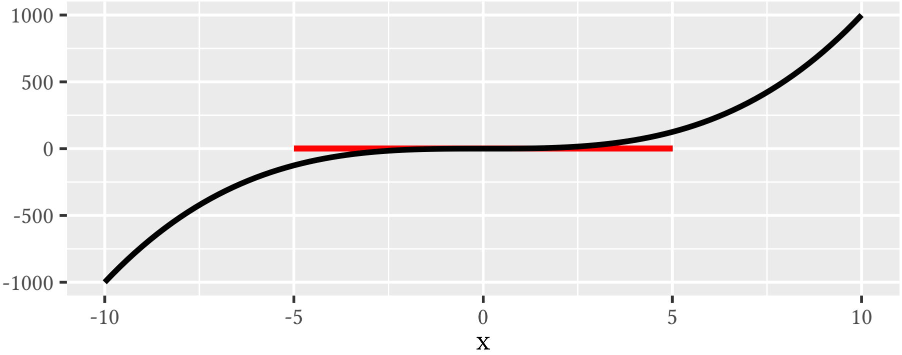
Minima: Example

$$f(x) = x^2$$



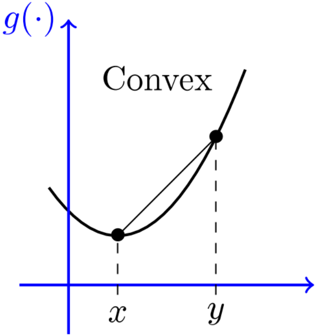
Saddle Point: Example

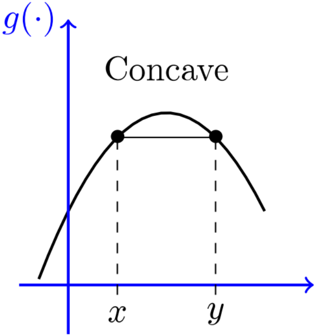
$$f(x) = x^3$$



Convex or Concave Functions

- The second derivative informs us how the function is bending (curvature)
- The second derivative, $f''(x)$, identifies whether the function f is concave or convex around x
  - If $f''(x) > 0 \Rightarrow$ Convex
  - If $f''(x) < 0 \Rightarrow$ Concave





Optimization

Extreme Value Theorem

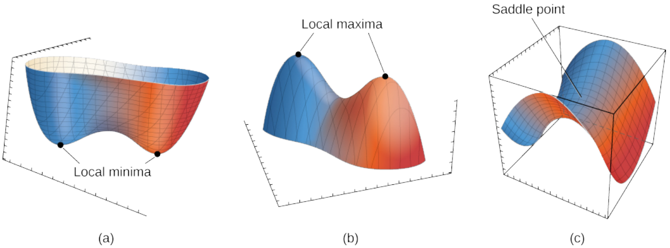
Suppose $f : [a, b] \to \mathbb{R}$ and that f is continuous. Then f obtains its extreme values (maximum and minimum) on $[a, b]$.

Corollary

Suppose $f : [a, b] \to \mathbb{R}$ is continuous and differentiable, and that neither $f(a)$ nor $f(b)$ is the extreme value. Then f obtains its extreme values on (a, b) and if $f(x_0)$ is the extreme value of f with $x_0 \in (a, b)$ then, $f^{'}(x_0) = 0$.

Solving Optimization

- Find $f^{'}(x)$ (First Order Condition)
  - Set $f^{'}(x) = 0$ and solve for x
  - Call all x_0 such that $f^{'}(x_0) = 0$ critical values
- Find $f^{''}(x)$ (Second Order Condition) and evaluate at each x_0
  - If $f^{''}(x_0) > 0$, Convex, local minimum
  - If $f^{''}(x_0) < 0$, Concave, local maximum
  - If $f^{''}(x_0) = 0$, Inconclusive, local minimum, maximum, or saddle point
- Check end points and compare them with local extremum



Optimization: Examples

- Find all maxima and minima:

$$f(x) = x^3 - 3x \text{ for } x \in [-2, 5]$$

Optimization: Examples

- Find all maxima and minima:

$$f(x) = \ln(x) - x \text{ for } x > 0$$